451-208 Computational Methods in Geomatics 2005
Topic 8 - Introduction to Rotation Matrices

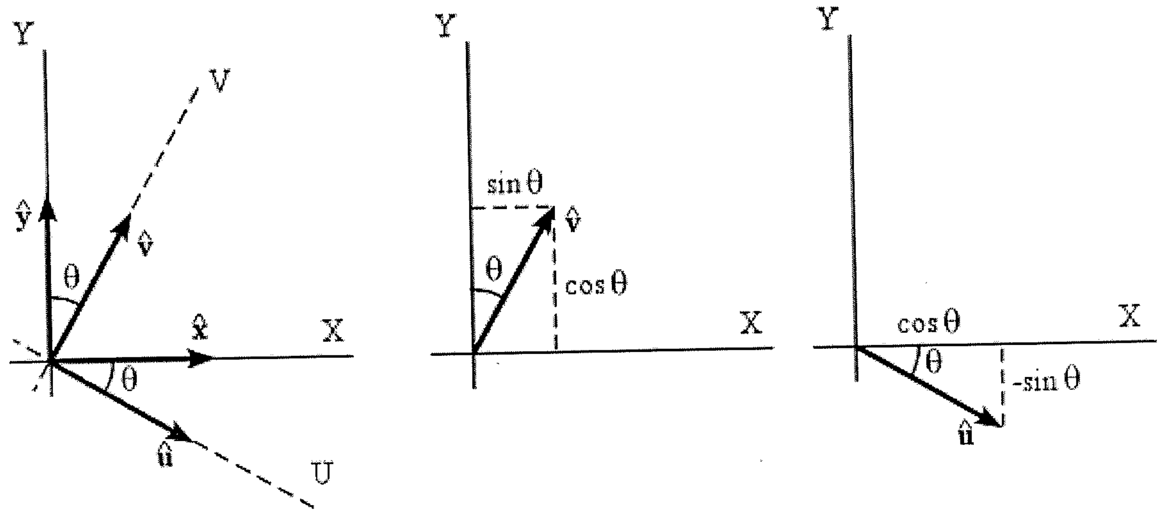

Figure 1

Figure-1 shows two reference frames: XY and UV . The UV reference frame has been rotated through the angle θ from the XY reference frame.

The unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ define the X-axis and Y-axis.
The unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ define the U-axis and V-axis.

The components of the unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in the XY reference frame are shown in Figure-1.

<u>Thus, in the XY reference frame</u> the unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ have the components:

$$ {}_{xy}\hat{\mathbf{u}} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \qquad {}_{xy}\hat{\mathbf{v}} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} $$

<u>In the UV reference frame</u> the unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ have the components:

$$ {}_{uv}\hat{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad {}_{uv}\hat{\mathbf{v}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} $$

R is the rotation matrix that transforms the vectors ${}_{uv}\hat{\mathbf{u}}$ and ${}_{uv}\hat{\mathbf{v}}$ in the UV system into ${}_{xy}\hat{\mathbf{u}}$ and ${}_{xy}\hat{\mathbf{v}}$ in the XY system

That is:

$$ {}_{xy}\hat{\mathbf{u}} = \mathbf{R}\,{}_{uv}\hat{\mathbf{u}} \qquad \text{and} \qquad {}_{xy}\hat{\mathbf{v}} = \mathbf{R}\,{}_{uv}\hat{\mathbf{v}} $$

Thus:

$$ \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} = \mathbf{R}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} = \mathbf{R}\begin{bmatrix} 0 \\ 1 \end{bmatrix} $$

These two equations can be combined as:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \mathbf{R}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Hence:} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

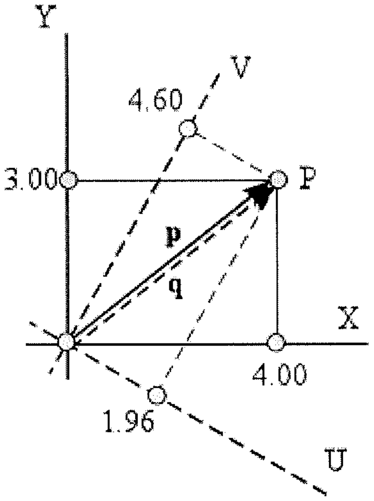

Figure 2

Consider the position vector of the point P in Figure-2. In the XY reference frame the position vector of P is **p**. In the UV reference frame the position vector is **q**.

Thus, in the XY system: $\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4.00 \\ 3.00 \end{bmatrix}$, in the UV system: $\mathbf{q} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1.96 \\ 4.60 \end{bmatrix}$

We have shown $\mathbf{p} = \mathbf{R}\mathbf{q}$

That is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \text{ which evaluates as } \begin{bmatrix} 4.00 \\ 3.00 \end{bmatrix} = \begin{bmatrix} .866 & .500 \\ -.500 & .866 \end{bmatrix}\begin{bmatrix} 1.96 \\ 4.60 \end{bmatrix}$$

As a general rule for constructing rotation matrices:
If the unit vectors ${}_1\hat{\mathbf{u}}$ and ${}_1\hat{\mathbf{v}}$ which define the axes of reference-frame-1 are represented by ${}_2\hat{\mathbf{u}}$ and ${}_2\hat{\mathbf{v}}$ in reference-frame-2, the rotation matrix to transform vectors in reference-frame-1 into reference-frame-2 is $[{}_2\mathbf{u} \quad {}_2\mathbf{v}]$

To transform a vector from the reference-frame-2 into reference-frame-1 we can use:

$$\mathbf{q} = \mathbf{R}^{-1}\mathbf{p}$$

But as **R** is **orthogonal,** $\boxed{\mathbf{R}^{-1} = \mathbf{R}^{T}}$ (see proof below)

Thus: $\mathbf{q} = \mathbf{R}^{T}\mathbf{p}$

That is:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ which evaluates as } \begin{bmatrix} 1.96 \\ 4.60 \end{bmatrix} = \begin{bmatrix} .866 & -.500 \\ .500 & .866 \end{bmatrix} \begin{bmatrix} 4.00 \\ 3.00 \end{bmatrix}$$

Note that in the discussion so far, it is considered that the axes rotate and the vector stays fixed.

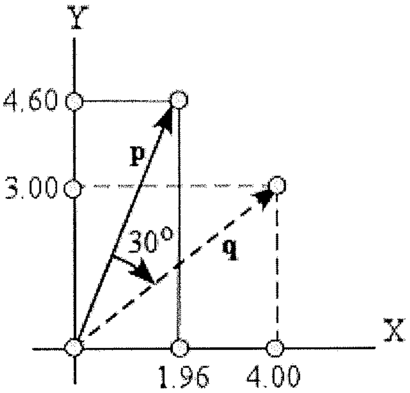

Figure 3

If we wish to see the effect of rotating the vector p through +30° within the XY axes system this can be considered the same as transforming a vector from the reference-frame-2 into reference-frame-1 by rotating the UV axes through an angle of -30°.

That is, we can use the rule $\mathbf{q} = \mathbf{R}^T\mathbf{p}$ with $\theta = -30^\circ$

However this can be simplified as $\mathbf{R}^T$ with and angle of -30° is the same as $\mathbf{R}$ with an angle of +30° as is shown below.

$$\mathbf{R}^T = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \mathbf{R}$$

as $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$

Thus, as a general rule:
To rotate the vector **p** through θ to **q** within the XY axes system:

$$\mathbf{q} = \mathbf{Rp} \qquad \text{where } \mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

For the case shown in Figure-3,

$$\mathbf{q} = \begin{bmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{bmatrix} \begin{bmatrix} 1.96 \\ 4.60 \end{bmatrix} \text{ which evaluates to } \begin{bmatrix} .866 & .500 \\ -.500 & .866 \end{bmatrix} \begin{bmatrix} 1.96 \\ 4.60 \end{bmatrix} = \begin{bmatrix} 4.00 \\ 3.00 \end{bmatrix}$$

That Rotation Matrices are Orthogonal

An orthogonal matrix is one where the transpose is the same as the inverse. All rotation matrices are orthogonal as is proved below.

The square of the magnitude of any vector $\mathbf{p}$ can be expressed (in matrix algebra) as $\mathbf{p}^T\mathbf{p}$.

$$\mathbf{p}^T\mathbf{p} = \begin{bmatrix} e & n & h \end{bmatrix} \times \begin{bmatrix} e \\ n \\ h \end{bmatrix} = (e^2 + n^2 + h^2)$$

Consider the vector $\mathbf{p}$ which is rotated as:

$$\mathbf{p} = \mathbf{R}^T\mathbf{p}$$

The square of the magnitude of $\mathbf{q}$ will be: $\mathbf{q}^T\mathbf{q}$

$$\mathbf{q}^T\mathbf{q} = (\mathbf{R}^T\mathbf{p})^T(\mathbf{R}^T\mathbf{p}) = \mathbf{p}^T\mathbf{R}\mathbf{R}^T\mathbf{p}$$

As the magnitude of the vector will not change with rotation;

$\mathbf{p}^T\mathbf{R}\mathbf{R}^T\mathbf{p}$ must equal $\mathbf{p}^T\mathbf{p}$

Thus:

$\mathbf{R}\mathbf{R}^T = \mathbf{I}$ (the identity matrix)

and hence $\mathbf{R}$ must be an orthogonal matrix as:

$$\mathbf{R}^{-1} = \mathbf{R}^T \qquad \text{and} \qquad \mathbf{R}^T\mathbf{R} = \mathbf{I}$$

If we consider the columns of rotation matrix $\mathbf{R}$ being made up with the vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ so that: $\mathbf{R} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix}$

$$\mathbf{R}^T\mathbf{R} = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \mathbf{c}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1^T\mathbf{c}_1 & \mathbf{c}_1^T\mathbf{c}_2 & \mathbf{c}_1^T\mathbf{c}_3 \\ \mathbf{c}_2^T\mathbf{c}_1 & \mathbf{c}_2^T\mathbf{c}_2 & \mathbf{c}_2^T\mathbf{c}_3 \\ \mathbf{c}_3^T\mathbf{c}_1 & \mathbf{c}_3^T\mathbf{c}_2 & \mathbf{c}_3^T\mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The vectors making up the columns in an orthogonal matrix must be unit vectors as their magnitudes ($\mathbf{c}^T{}_1\mathbf{c}_1$ for example)are 1:

$$\mathbf{c}^T{}_1\mathbf{c}_1 = 1 \qquad \mathbf{c}^T{}_2\mathbf{c}_2 = 1 \qquad \mathbf{c}^T{}_3\mathbf{c}_3 = 1$$

Also, the column vectors must be orthogonal as the dot products are zero:

$$\mathbf{c}^T{}_1\mathbf{c}_2 = 0 \qquad \mathbf{c}^T{}_1\mathbf{c}_3 = 0 \qquad \mathbf{c}^T{}_2\mathbf{c}_3 = 0$$

By a similar argument it can be shown that the vectors making up the rows of a rotation matrix are unit orthogonal vectors.

Transformation of vectors between Geocentric and Local frames of reference

In Figure-1, the XYZ axes system has an origin at the centre of the reference ellipsoid, the Z-axis points to the North Pole, and the X-axis is the intersection of the planes of the meridian through Greenwich and the equator. The Y-axis makes up a right handed orthogonal set. These axes define the **geocentric** reference frame.

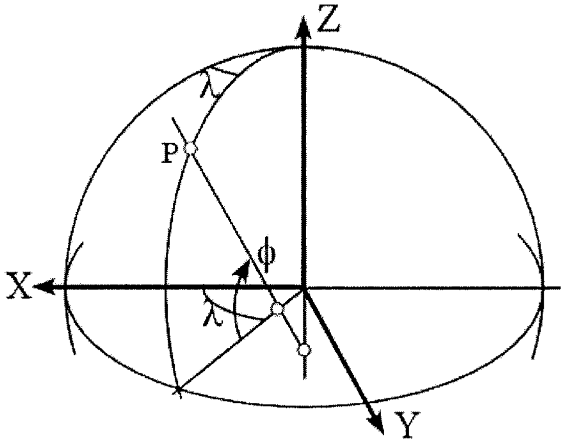

Figure 1

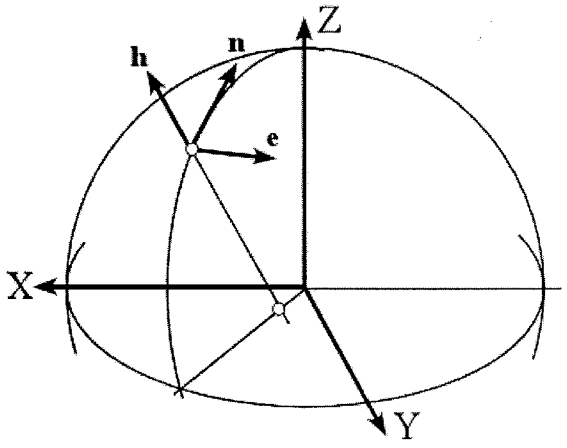

Figure 2

P is a point with longitude λ and latitude ϕ . A local reference frame has its origin at P and is defined by the three mutually orthogonal (local) unit vectors $\mathbf{e}_l$, $\mathbf{n}_l$ and $\mathbf{h}_l$.

Thus in the local reference frame:

$$\mathbf{e}_l = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{n}_l = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{h}_l = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In the geocentric reference frame these vectors can be written as:

$$\mathbf{e}_g = \begin{bmatrix} \delta x_e \\ \delta y_e \\ \delta z_e \end{bmatrix} \qquad \mathbf{n}_g = \begin{bmatrix} \delta x_n \\ \delta y_n \\ \delta z_n \end{bmatrix} \qquad \mathbf{h}_g = \begin{bmatrix} \delta x_h \\ \delta y_h \\ \delta z_h \end{bmatrix}$$

It can be shown (see later in notes) that:

$$\mathbf{e}_g = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix} \qquad \mathbf{n}_g = \begin{bmatrix} -\sin\phi\cos\lambda \\ -\sin\phi\sin\lambda \\ \cos\phi \end{bmatrix} \qquad \mathbf{h}_g = \begin{bmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{bmatrix}$$

Also, vectors in one reference frame can be transformed into another by multiplication by a rotation matrix.

Thus:

$$\mathbf{e}_g = \mathbf{R}\,\mathbf{e}_l \qquad \mathbf{n}_g = \mathbf{R}\,\mathbf{e}_l \qquad \mathbf{n}_g = \mathbf{R}\,\mathbf{e}_l$$

or $$\mathbf{e}_g = \mathbf{R}\begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \mathbf{n}_g = \mathbf{R}\begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \mathbf{h}_g = \mathbf{R}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

These expression can written in full as

$$\begin{bmatrix}-\sin\lambda\\ \cos\lambda\\ 0\end{bmatrix} = \begin{bmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} \qquad (1)$$

$$\begin{bmatrix}-\sin\phi\cos\lambda\\ -\sin\phi\sin\lambda\\ \cos\phi\end{bmatrix} = \begin{bmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} \qquad (2)$$

$$\begin{bmatrix}\cos\phi\cos\lambda\\ \cos\phi\sin\lambda\\ \sin\phi\end{bmatrix} = \begin{bmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} \qquad (3)$$

The terms in the rotation matrix can be determined by evaluating equation (1), (2) and (3)

For example, in equation (1), multiplying the top row of $\mathbf{R}$ into the vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$ gives:

$$-\sin\lambda = r_{11} * 1 + r_{22} * 0 + r_{13}*0 = r_{11}$$

Similarly, multipltying the 2nd and 3rd row:

$$\cos 1 = r_{21} \qquad 0 = r_{31}$$

Similar action on equations (2) and (3) gives:

$$r_{12} = -\sin\phi\cos\lambda \qquad r_{22} = -\sin\phi\sin\lambda \qquad r_{32} = \cos\phi$$

$$r_{13} = \cos\phi\cos\lambda \qquad r_{13} = \cos\phi\sin\lambda \qquad r_{13} = \sin\phi$$

Thus rotation matrix R can be written as:

$$\mathbf{R} = \begin{bmatrix}-\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda\\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda\\ 0 & \cos\phi & \sin\phi\end{bmatrix}$$

This demonstrates that the columns of the rotation matrix to transform a vector from the local to the geocentric reference frame are the vectors:

$$\mathbf{e}_g = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix} \qquad \mathbf{n}_g = \begin{bmatrix} -\sin\phi\cos\lambda \\ -\sin\phi\sin\lambda \\ \cos\phi \end{bmatrix} \qquad \mathbf{h}_g = \begin{bmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{bmatrix}$$

The rotation matrix developed is one that transforms a vector in the local reference frame into a vector in the geocentric reference frame. If $\mathbf{l}$ is a vector in the local reference frame and $\mathbf{g}$ its representation in the geocentric reference frame:

$$\mathbf{g} = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}, \quad \mathbf{l} = \begin{bmatrix} \delta e \\ \delta n \\ \delta h \end{bmatrix}, \qquad \mathbf{g} = \mathbf{Rl}$$

In detail,

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} \delta e \\ \delta n \\ \delta h \end{bmatrix}$$

As $\mathbf{R}$ is orthogonal:

$$\mathbf{l} = \mathbf{R}^T\mathbf{g}$$

In detail,

$$\begin{bmatrix} \delta e \\ \delta n \\ \delta h \end{bmatrix} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$$

Example-1

A traverse line in a local datum at latitude -38° and longitude 145° has bearing 30°, elevation 10° and length 1000.000 metres

$$\text{Local vector } \mathbf{l} = 1000\begin{bmatrix} \sin 30 \cos 10 \\ \cos 30 \cos 10 \\ \sin 10 \end{bmatrix} = \begin{bmatrix} 492 \\ 853 \\ 174 \end{bmatrix}$$

$$\text{Geocentric vector } \mathbf{g} = \mathbf{Rl} = \begin{bmatrix} -\sin(145) & -\sin(-38)\cos(145) & \cos(-38)\cos(145) \\ \cos(145) & -\sin(-38)\sin(145) & \cos(-38)\sin(145) \\ 0 & \cos(-38) & \sin(-38) \end{bmatrix}\begin{bmatrix} 492 \\ 853 \\ 174 \end{bmatrix}$$

$$\mathbf{g} = \begin{bmatrix} -.574 & -.504 & -.646 \\ -.819 & .353 & .452 \\ .000 & .788 & -.616 \end{bmatrix}\begin{bmatrix} 492 \\ 853 \\ 174 \end{bmatrix} = \begin{bmatrix} -825 \\ -24 \\ 565 \end{bmatrix}$$

Example-2

The geocentric coordinates for the first order control stations at Kangaroo Ground and Mt Dandenong are approximately:

$$\begin{bmatrix} -4149858 \\ 2882904 \\ -3879343 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -4149979 \\ 2868064 \\ -3890828 \end{bmatrix}$$

Thus the vector in the geocentric reference frame from Kangaroo Ground to Mt Dandenong is:

$$\mathbf{g} = \begin{bmatrix} -121 \\ -14840 \\ -11485 \end{bmatrix}$$

The vector in the local reference frame will be:

$$\mathbf{l} = \mathbf{R}^T\mathbf{g} = \begin{bmatrix} -.574 & -.819 & .000 \\ -.504 & .353 & .788 \\ -.646 & .452 & -.616 \end{bmatrix}\begin{bmatrix} -121 \\ -14840 \\ -11485 \end{bmatrix} = \begin{bmatrix} 12226 \\ -14230 \\ 441 \end{bmatrix}$$

The local vector is length 18765.7m, bearing 139.2°, elevation 1.3°

Developing the components of $\mathbf{e}_g$, $\mathbf{n}_g$, and $\mathbf{h}_g$

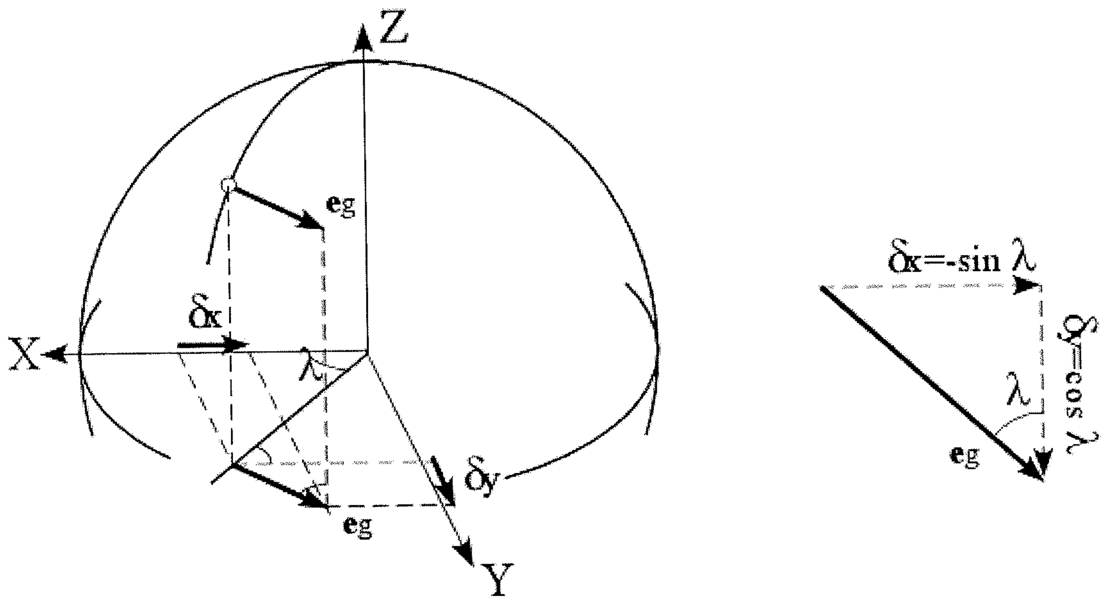

Figure 3

Figure 3 shows the construction needed to show that the vector $\mathbf{e}_g$ has components in the geocentric reference frame as:

$$\mathbf{e}_g = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix}$$

The construction has been made with the origin of the local system (point P) in the northern hemisphere (positive latitude) and with a longitude between 0° and 90°. In this case the bearing of $\mathbf{e}_g$ is such that δx will be negative and δy positive.

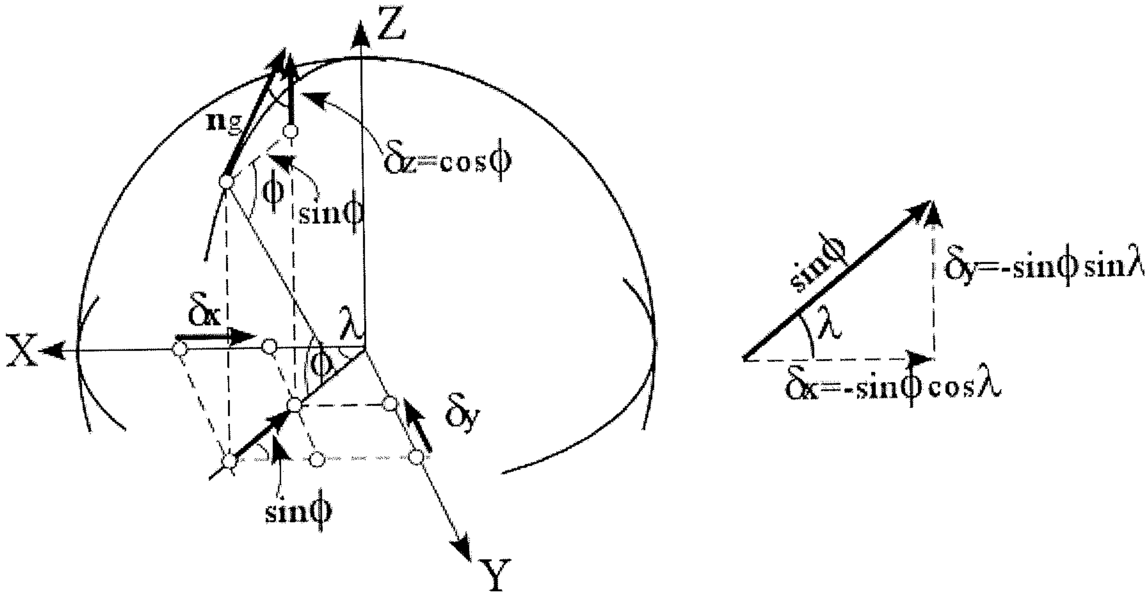

Figure 4

Figure-4 shows the projections of the unit vector defining the north axis $\mathbf{n}_g$ onto the X, Y and Z axis – the axes of the geocentric reference frame.

From this it can be seen that the vector has components:

$$\mathbf{n}_g = \begin{bmatrix} -\sin\phi\cos\lambda \\ -\sin\phi\sin\lambda \\ \cos\phi \end{bmatrix}$$

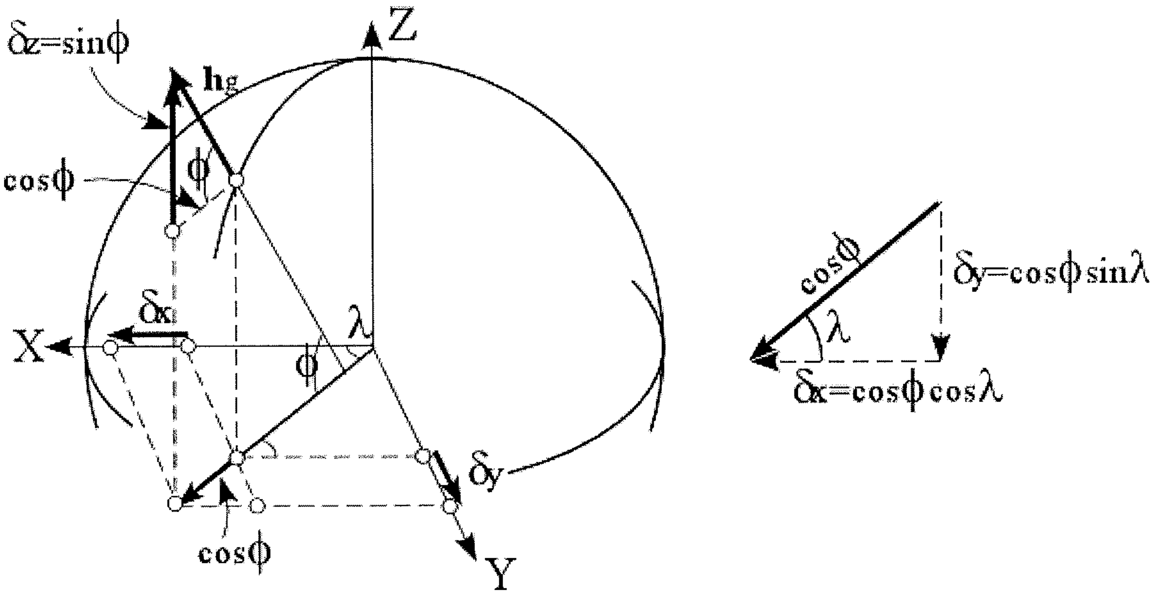

Figure 5

Figure-5 shows the projections of the unit vector defining the height axis $\mathbf{h}_g$ onto the X, Y and Z axis – the axes of the geocentric reference frame.

From this it can be seen that the vector has components:

$$\mathbf{h}_g = \begin{bmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{bmatrix}$$